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Perturbed turbulent flow, eddy viscosity and the generation of turbulent stresses

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The perturbation of a turbulent flow by an organized wavelike disturbance is examined using a dynamical, rather than phenomenological, approach. On the basis of the assumption that an infinitesimal perturbation results in a linear change in the statistics of the turbulence, and that the turbulence is either weak or that the turbulent perturbations are quasi-Gaussian, a method for predicting the perturbation turbulent Reynolds stresses is developed. The novel aspect of the analysis is that all averaging is delayed until the dynamical equations have been solved rather than attempting to find, *a priori*, equations for averaged quantities. When applied to long-wave perturbations the analysis indicates that the perturbation shear stress is of primary dynamical importance, and that this stress is determined by the principal component of mean shear through a relation which depends on the spectrum of the turbulent velocity component parallel to the gradient of the undisturbed mean velocity (the component perpendicular to the wall in a turbulent boundary layer). Theoretical arguments and observations are used to estimate the form of this spectrum in a constant stress shear layer. This results in a prediction of the constitutive law relating turbulent stress and the mean flow. The law is visco-elastic in nature, and is in agreement with the known constitutive relation for stress perturbations to a constant stress boundary layer; it resembles the eddy viscosity relation used successfully by others in describing perturbations in turbulent flows. The details of the constitutive law depend on how well the turbulence obeys Taylor's hypothesis that phase velocity equals mean flow velocity, and some insight into this question is given.

1. Introduction

The prediction of the mean velocity and momentum fluxes in turbulent flow is a central and persistent problem in fluid mechanics. A special aspect of this subject which has received considerable attention in recent years is the prediction of the behaviour of small-amplitude coherent perturbations introduced into a turbulent flow with a parallel mean velocity and a known statistical description of the turbulent component. Interest in this area stems, in part, from the intrinsic importance of particular problems of this type, such as the determination of the stability of the mean component of turbulent flows and the prediction of the mean description of turbulent flow over a wave. But, from a more general point of view, these problems are particularly interesting because they represent a uniquely attractive context in which to study the dynamics of turbulent flows. This attractiveness stems from the significant simplification to both experimental and theoretical studies resulting from the linearity of sufficiently small perturbations.

The problems of interest here may be described in terms of the following experiments. Suppose we begin with a flow apparatus capable of producing a turbulent flow in which the mean flow velocity $\mathbf{U}$ is in the x_1 direction and has the magnitude $U(x_3)$; similarly all the statistics of the turbulent velocity depend only on x_3. This 'parallel in the mean' flow will be called the primary or undisturbed flow. The constant stress boundary layer or fully developed turbulent flow in a pipe are examples of such parallel in the mean primary flows. Now suppose the experimenter is able to introduce into his apparatus some deterministic infinitesimal disturbance which, in a mathematical sense, would be a prescribed perturbation of the boundary conditions. This perturbation might, for example, be a small-amplitude wave on one of the lateral boundaries, or a vibrating ribbon upstream of the measurement area. We are interested in predicting the mean flow in the perturbed condition, assuming the complete statistical description of the primary state is known. The term 'mean' is defined as the average obtained from a large number of similarly prepared realizations of the perturbed flow in all of which the imposed boundary perturbations are identical. Operationally this is accomplished by adopting, in each realization, a co-ordinate system x_1, x_2, x_3, t, such that the imposed boundary perturbations all have the same description. Then the mean value $\langle g(\mathbf{x}, t)\rangle$ is the average over many realizations of the value of g at the fixed point $(\mathbf{x}, t)$.

In each realization of the primary flow, the velocity is $\mathbf{U}(x_3)+\mathbf{u}(x,t)$, where the mean of the turbulent component $\mathbf{u}$ is zero. In each realization of the perturbed flow, the velocity is $\mathbf{U}(x_3)+\langle\mathsf{U}\rangle+\mathbf{u}'$. Momentum equations describing the mean flow can be obtained by the usual technique of averaging the Navier–Stokes equations; these equations involve the turbulent Reynolds stresses, which are $\langle -\rho u_n u_m\rangle$ and $\langle -\rho u_n' u_m'\rangle$ in the primary and perturbed flows respectively. If $\langle\mathsf{U}\rangle$ is sufficiently small, a linear equation for the mean perturbation (see Hussain & Reynolds 1972*b*; Davis 1972)

$$(\partial t + U\partial_1)\langle \mathsf{U}_n\rangle + \delta_{1n} U'\langle \mathsf{U}_3\rangle + \partial_n\langle \mathsf{P}\rangle = \partial_m \mathsf{R}_{nm} \tag{1}$$

may be obtained by subtracting the equation for the primary flow from the equation describing the perturbed flow. Here $\langle\mathsf{P}\rangle$ is the mean perturbation pressure divided by density, and $\mathsf{R}_{nm} = \langle -u_n' u_m'\rangle - \langle -u_n u_m\rangle$ is the density normalized perturbation turbulent Reynolds stress.

The prediction of the perturbation turbulent stress R_{nm} is of course the central difficulty in deducing the dynamics of perturbations in turbulent flows, and is the subject of this paper. To date, all attempts to deduce such dynamics have employed phenomenological laws of various degrees of complexity to predict the turbulent stresses. One of the most fascinating results obtained from the comparison of experimental measurements and the predictions of such theoretical studies is the surprising success of predictions based on an eddy viscosity relation between the stresses R_{nm} and the mean rate of strain $\partial_m\langle\mathsf{U}_n\rangle$. Both Hussain

& Reynolds (1972*b*), who were investigating wavelike disturbance introduced into a fully developed turbulent channel flow by a vibrating ribbon, and Davis (1972), who was investigating perturbation of a turbulent boundary layer by a wavelike disturbance of the boundary, obtained encouraging, if not conclusive, results using the scalar eddy viscosity that is consistent with the undisturbed flow.

While the success of these eddy viscosity models is tantalizing, it fails to be completely satisfying in two ways. (i) The comparison of theory and experiment is indirect, in the sense that the perturbation Reynolds stresses are not compared directly to the rate of strain. Rather, selected properties of the mean flow are compared with dynamical solutions of the equations of motion obtained by assuming a viscous relationship between stress and rate of strain. (ii) Perhaps more unsatisfactory is the lack of any plausible dynamical explanation of why the phenomenologically motivated eddy viscosity concept should apply to turbulent flows. Apart from the highly idealized quasi-molecular picture leading to mixing length theories, there does not seem to be any physical model that leads to a viscous constitutive relation for turbulent fluids. In fact, Townsend (1966), Crow (1968), Lumley (1970) and others suggest that in many ways a turbulent flow more nearly resembles an elastic medium.

Here a highly simplified dynamical model is used to investigate the generation of Reynolds stresses by an infinitesimal perturbation of a turbulent shear flow. The philosophy is that a dynamical description, even one based on very severe mathematical approximations, will lead to a more reliable form for the constitutive equation of a turbulent fluid than can be obtained by phenomenological analogies with processes of uncertain relevance to turbulence. The objective of this study is thus to determine the general structure, if not a completely quantitative form, of the constitutive relationship between the mean velocity and the mean Reynolds stresses in a perturbed turbulent shear flow.

The dynamical model employed is based on several assumptions and simplifications. Some of these are mathematical and are introduced only to simplify the analysis to the point where a solution can be obtained without extensive numerical computation. Certain assumptions are necessitated by the lack of a sufficiently detailed experimental description of the kinematics of the undisturbed turbulent shear flow, and these assumptions can be verified or corrected only by more measurement. But, there are two fundamental assumptions, essential to the entire approach, which should therefore be emphasized at the outset. But before proceeding to a discussion of these it is necessary to introduce certain definitions.

Let the velocity associated with a single turbulent flow realization with undisturbed boundary conditions be $\mathbf{U}+\mathbf{u}$, where $\mathbf{U}$ is the mean of a large number of such realizations. If this flow is described by the deterministic Navier–Stokes equation, then, in principle at least, this flow could be reproduced, given perfect control of the initial and boundary conditions. The fundamental assumption of the model introduced here is that, if this flow were reproduced in the presence of an infinitesimal boundary perturbation, then, in the observed velocity $\mathbf{U}+\mathbf{u}+\boldsymbol{\mathsf{U}}$, the perturbation velocity $\boldsymbol{\mathsf{U}}$ would be, in a statistical sense, small. The deviation from the unperturbed flow is $\boldsymbol{\mathsf{U}}$, which has both a mean part $\langle\boldsymbol{\mathsf{U}}\rangle$ and a turbulent part; the velocity $\mathbf{u}+\boldsymbol{\mathsf{U}}$ is equal to $\mathbf{u}'+\langle\boldsymbol{\mathsf{U}}\rangle$ of the previously given example.

The statistically small assumption is required to arrive at linear dynamical equations for the perturbation velocity. The validity of the assumption does not require that $\mathbf{U}$ itself be small, only that it enter linearly into the statistics of the perturbed flow, so that, using a loose notation for order of magnitudes,

$$U\langle \mathsf{U}\rangle \gg \langle \mathsf{UU}\rangle, \quad \langle uu\mathsf{U}\rangle \gg \langle u\mathsf{UU}\rangle. \tag{2a}$$

There might be a serious philosophical question whether this assumption is correct, since we normally consider turbulent flows to be unstable, in the sense that an infinitesimal perturbation results in an order unity change of the flow. This idea is not, however, inconsistent with the 'statistically small' assumption; but it does not raise the question of its validity. Unfortunately, there seems to be no straightforward way to check this hypothesis; thus one is forced to rely on indirect tests of the predictions made using the assumption. It should be pointed out, however, that the hypothesis appears to be self-consistent, in that the perturbation velocities obtained by accepting the hypothesis are statistically small if the boundary perturbation is small. It should also be noted that this assumption is equivalent to the apparently reasonable hypothesis that the perturbation stress R is linearly related to the amplitude of the imposed boundary perturbation.

The second major assumption involved in this development concerns averages of the third-order tensor $u_n u_m \mathsf{U}_l$:

$$\langle u_n u_m \mathsf{U}_l\rangle \simeq \langle u_n u_m\rangle \langle \mathsf{U}_l\rangle. \tag{2b}$$

This will be true in either of two cases. (i) It will be true if $\mathbf{u}$ and $\mathbf{U}$ have approximately jointly Gaussian probability distributions. (ii) It will be true if the turbulence is weak, in the sense that $\langle u^2\rangle \ll U^2$. This is obvious in the limiting case of a laminar flow, where $\mathbf{U} = \langle \mathbf{U}\rangle$. It will be seen in §2 that this approximation remains formally correct so long as (using a loose notation for orders of magnitude) $u \ll U$.

The structure of the model resulting from these assumptions is such that the predictions of the mean perturbation velocity $\langle \mathbf{U}\rangle$ and the mean turbulent Reynolds stress tensor depend on somewhat detailed knowledge of the complete wavenumber–frequency spectrum of the primary turbulent velocity $\mathbf{u}$. There exists a considerable body of data concerning the frequency spectrum of turbulent shear flows, particularly for the very important example of a constant stress boundary layer. Unfortunately, the available data are not adequate to allow determination of the wavenumber spectrum. For example, most descriptions of the downstream structure of the turbulence are based on Taylor's (1938) hypothesis that the frequency ω of a turbulent component is equal to $\mathbf{U}.\mathbf{k}$, where $\mathbf{k}$ is the vector wavenumber. Taylor's hypothesis, is of course, an approximation, and is not expected to be precise, either when the turbulent intensity is large, or when $\mathbf{U}$ is not spatially uniform. It is expected that the spectrum at a given wavenumber will be peaked near the frequency $\mathbf{U}.\mathbf{k}$, but this peak will have a finite width $\Delta\omega$. On dimensional grounds, it is expected that the bandwidth $\Delta\omega$ will be related to the frequencies $\langle u^2\rangle^{\frac{1}{2}}k$ and $|\nabla.\mathbf{U}|$. Using the data of Morrison & Kronauer (1969), a quantitative estimate of $\Delta\omega$ for a constant stress boundary layer will be made and used to find an approximate description of the required features of the horizontal structure of the turbulence. With regard to the vertical structure of the turbulence, the situation is even more uncertain, and I feel this results in the greatest limitations on the present theory.

The model developed here results in a functional relation between $\langle \mathbf{U} \rangle$ and the turbulent Reynolds stress tensor R, which is visco-elastic and involves no adjustable constants. It is encouraging that the predictions are, within the uncertainty of the data on spectra of $\mathbf{u}$, in agreement with the known relation between mean flow and stress associated with the 'law of the wall'.

2. The dynamical model

As discussed in §1, the dynamical model considered here is based on a hierarchy of simplifications, some fundamental and others required for computational economy or necessitated by the lack of a complete statistical description of the primary turbulence. The fundamental assumptions (2) allow development of a formally exact method of describing the perturbed mean flow. The general structure of the model is novel, and deserves exposition by itself, without consideration of the additional simplifications required to make it operationally useful. In §2 the problem of relating the turbulent stress R to the mean velocity $\langle \mathbf{U} \rangle$ is posed mathematically and the general method of solution is outlined formally. Application of the method is considered in §§3 and 4.

From the definitions of the velocity components (§1), it can be seen that the perturbation component obeys the continuity relation

$$\partial_n \mathsf{U}_n = 0. \tag{3a}$$

If we consider a parallel undisturbed velocity U along x_1 which varies only with x_3 then the perturbation momentum equation is

$$(\partial_t + U\partial_1)\,\mathsf{U}_n + \delta_{1n}\,U'\mathsf{U}_3 + \partial_n \mathsf{P} - \nu\partial_m\partial_m \mathsf{U}_n = -\partial_m(u_m \mathsf{U}_n + \mathsf{U}_m u_n) + \mathsf{S}_n, \tag{3b}$$

where S_n involves products of the form UU and P is the perturbation pressure divided by the fluid density. Here interest is restricted to flow past smooth walls, so that

$$\mathsf{U}_n = 0 \quad \text{at} \quad x_3 = H_0, H_1. \tag{3c}$$

This is appropriate, for example, for the vibrating-ribbon experiment of Hussain & Reynolds (1970, 1972*a*), but not for flow over a wave. This latter, more complicated, case will be considered in a later paper.

Let $\mathbf{W}$ be the vector with components $[\mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3, \mathsf{P}]$. Then (3) can be written symbolically as

$$L_{nm}\mathsf{W}_m = \mathsf{Q}_n, \tag{3d}$$

where the rows $n = 1, 2, 3$ are (3*b*) with

$$\mathsf{Q}_n = -\partial_m(\mathsf{U}_n u_m + \mathsf{U}_m u_n) + \mathsf{S}_n.$$

The row $n = 4$ is (3*a*) with $\mathsf{Q}_4 = 0$. The boundary conditions (3*c*) provide six more rows of the operator L_{nm} with $\mathsf{Q}_n = 0$.

The quantity of eventual interest is the mean flow described by $\langle \mathsf{W} \rangle$. Averaging ($3d$) gives

$$L_{nm}\langle \mathsf{W}_m \rangle = D_{nk}\,\partial_m \mathsf{R}_{km}, \tag{4}$$

where $D_{nm} = \delta_{nm}$ for $n = 1, 2, 3$ and zero otherwise; R_{nm}, the same quantity as appears in (1), is related to the velocity components through

$$\mathsf{R}_{nm} = \langle -u_n \mathsf{U}_m \rangle + \langle -u_m \mathsf{U}_n \rangle.$$

The first three rows of (4) are identical to (1), except for notation and the inclusion of the direct viscous effects retained in the operator L.

Historically, two basic approaches have been employed to predict R. The first is to assume a phenomenological constitutive relation such as the eddy viscosity law of Hussain & Reynolds (1972b) or the more complicated eddy visco-elasticity law of Davis (1972). An apparently more sophisticated approach is to multiply ($3b$) by u_k, then average. This results in an underdetermined system of Reynolds stress conservation equations, which requires introduction of additional phenomenological relations connecting the unknowns $\langle \mathsf{W} \rangle$ and R to the additional variables $\langle u_n \mathsf{P} \rangle$ and $\langle u_n \partial_m \partial_m \mathsf{U}_k \rangle$ that appear. This method was found to be unsuccessful by Davis (1972).

The purpose of this paper is to introduce a method of predicting R which avoids the phenomenology of these earlier approaches. The novel feature is that all averaging is delayed until the dynamical equations have been solved, which is to be contrasted with the usual procedure of attempting to determine a set of equations which describes the average of the various relevant flow variables.† The technique of delaying the information-losing step of averaging eliminates the need for phenomenological relations between averages of flow variables, since the equations of motion determine completely the individual realizations of these variables.

The differential equation and boundary conditions represented by ($3d$) are all linear and, consequently, the operator L has an inverse such that the solution to the differential system can be represented as

$$\mathsf{U}_n(\mathbf{x}, t) = L^{-1}_{nk} \mathsf{Q}_k(\mathbf{y}, r) \quad (n = 1, 2, 3).$$

The operator L^{-1} involves an integral over the space variable $\mathbf{y}$ and time variable r of the product of $\mathsf{Q}(\mathbf{y}, r)$ and the appropriate Green's function. The variables $\mathbf{x}, t$ appear only parametrically in L^{-1}_{nk}. If this equation is multiplied by $u_m(\mathbf{x}, t)$, then u_m may be passed through the operator since, so far as the active variables $\mathbf{y}$ and r are concerned, u_m is a constant. Since the operator L is deterministic, the process of averaging also passes through the inverse, so that

$$\langle u_m \mathsf{U}_n \rangle (\mathbf{x}, t) = L^{-1}_{nk} \langle u_m(\mathbf{x}, t) \mathsf{Q}_k(\mathbf{y}, r) \rangle \quad (n = 1, 2, 3).$$

(It is the step of passing the average through L^{-1} that prevents this method from being applied directly to flow over a wavy boundary, because the wavy-wall

† I have recently become aware that the general method is similar to one used in the study of wave propagation in random media, and is a first approximation to a powerful technique developed by Keller (1964).

boundary conditions in L involve the turbulent velocities and consequently neither L nor its inverse is deterministic.)

The above expression for $\langle u_n \mathsf{U}_m \rangle$ is greatly simplified by the basic assumptions (2). Thus (2a) states that S, which appears in Q, can be neglected, and (2b) allows the average of the triple products $\langle uu\mathsf{U} \rangle$ to be simplified so that

$$\langle u_m \mathsf{U}_n \rangle = -L^{-1}_{nk} D_{kj} \frac{\partial}{\partial y_l} (\langle u_m u'_l \rangle \langle \mathsf{U}'_j \rangle + \langle u_m u'_j \rangle \langle \mathsf{U}'_l \rangle), \tag{5}$$

where unprimed quantities are evaluated at $\mathbf{x}$ and t, while primed quantities are functions of $\mathbf{y}$ and r, the integration variables of L^{-1}; here D has the same meaning as in (4).

While the complex notation tends to obscure it, (5) has a simple and interesting physical interpretation. Examination of this equation allows isolation of that component of the perturbation velocity $\mathbf{U}$ which, according to the approximations employed, gives rise to the perturbation Reynolds stresses $\langle u_n \mathsf{U}_m \rangle$. Thus we may define an 'active' turbulent velocity component $\hat{\mathbf{U}}$ by

$$L_{mn} \hat{\mathsf{W}}_m = -D_{mk} \partial_l (u_l \langle \mathsf{U}_k \rangle + u_k \langle \mathsf{U}_l \rangle), \tag{6}$$

which replaces (3d). From (5) it follows that

$$\langle u_n \mathsf{U}_m \rangle = \langle u_n \hat{\mathsf{U}}_m \rangle.$$

Clearly, the active turbulent component $\hat{\mathbf{U}}$ is that part of $\mathbf{U}$ which appears in the turbulent Reynolds stress in (5). Mechanically it is obtained by neglecting in (3) the contributions to Q of products of the forms

$$\mathsf{UU} \quad \text{and} \quad u(\mathsf{U} - \langle \mathsf{U} \rangle).$$

It is important to note, however, that the validity of (6) does not require that these quantities be small in any sense other than that implied by the fundamental 'statistically small' assumptions (2).

From (5) the major difficulties involved in computing the perturbation Reynolds stresses are easily seen. (i) A very complete statistical description of the primary turbulence field is required; the quantities $\langle uu' \rangle$ are two-point, two-time covariances of the undisturbed turbulence, which is not stationary in the x_3 direction. (ii) Construction of the inverse operator L^{-1} is difficult, and in general could be accomplished only through extensive numerical computation; even so, a different inverse operator would be required for every Reynolds number of the primary flow. As will be seen in §3, these difficulties can, to some extent, be overcome in the special case of long-wave perturbations.

3. Long-wave approximation

The general method outlined in §2 is, in principle, capable of producing a constitutive relation which relates the mean turbulent stresses $\langle u_n \mathsf{U}_m \rangle$ to the mean flow $\langle \mathsf{U}_n \rangle$. But, unfortunately, putting this general method into practice requires a description of the undisturbed turbulence which is considerably more detailed than anything that can be obtained from presently available measurements. Further application of the method without approximation would result in a constitutive relation of such complexity as to be nearly unusable. One is led, then, to the conclusion that there is no simple constitutive relation, even for the highly idealized circumstances under consideration.

Rather than abandon further examination in recognition of the impossibility of finding an exact and useful result, I have chosen to make some strong mathematical approximations which lead to a tractable analysis, and produce some simple results closely connected with the phenomenologically motivated concept of an eddy viscosity. The most serious of these approximations involves the restriction that both the primary turbulence and the perturbation velocities are composed of 'long' waves, in the sense that they vary much more rapidly in the x_3 direction than in the x_1, or downstream, direction. This is a reasonable, and not unduly constrictive restriction to place on the mean perturbation $\langle \mathsf{U} \rangle$. But, when applied to the primary turbulence $\mathbf{u}$, the long-wave approximation is not so much a restriction as an assumption about a prescribed quantity, and, unfortunately, it appears from the available data that it is not a very accurate assumption. While it is true that the energetic components of turbulent shear flows do have somewhat longer scales in the downstream than in the x_3 direction, the distinction is not clear-cut, and certainly not sufficient to warrant application of an asymptotic expansion in this ratio.

There is, however, a somewhat subtle reason for believing that the long-wave approximation may be much more accurate than a straightforward scale analysis would suggest. As will be seen, the approximation leads to simplified forms of (3d) in which only the x_3 derivative of the horizontal velocities U_1 and U_2 are retained on the right-hand side. These velocities may be considered as the sum of Fourier modes with downstream wavenumber k_1 and angular frequency ω. It will be seen that each of these modes has very large shears near its critical height, where $Uk_1+\omega = 0$, and, consequently, when the term

$$\mathsf{Q}_n = -\partial_m(\mathsf{U}_n u_m + \mathsf{U}_m u_n)$$

is considered as the sum of the interactions between u_n and each of these modes, it is seen that the retained term $u_3 \partial_3 \mathsf{U}_n$ is by far the largest at least in the neighbourhood of the mode's critical level. This assertion can be evaluated only by careful examination of the details of the analysis to be presented.

In this section we consider spatially and temporally periodic perturbations of a constant stress boundary layer described, for $z \gg z_0$, by

$$U = U_0 \ln(z/z_0),$$

where $z = x_3$. The mean perturbation $\langle \mathsf{W}_n \rangle$ is restricted to have no x_2 dependence and to propagate parallel to the x_1 axis. The region near the surface where viscosity has a direct influence (the viscous sublayer) is ignored, and it is assumed that the viscosity is sufficiently small that the limit $\nu \to 0$ is appropriate. In keeping with the discussion leading to (5), the perturbation velocity U_n is taken to vanish at $z = 0$ and as $z \to \infty$.

The velocities may be considered as made up of Fourier modes of the form

$$[\mathsf{U}_n, u_n] = \sum_{\mathbf{k}} [\mathsf{V}_n, v_n] \exp\{i\mathbf{k}.\mathbf{s}\}, \tag{7}$$

where $\mathbf{k}$ and $\mathbf{s}$ are the pseudo-vectors $[\omega, k_1, k_2]$ and $[t, x_1, x_2]$, respectively. The mean perturbation $\langle \mathsf{U}_n \rangle$ is associated with the wavenumbers $\pm \mathbf{k}_0$, where

$$\mathbf{k}_0 = [\omega_0, k_0, 0].$$

The mean perturbation is taken to be 'long' in the sense that $k_0 z_c \ll 1$, where z_c is the critical height at which $Uk_0 + \omega_0 = 0$. The immediate consequences of this restriction are as follows.

(i) The turbulent stresses $\langle u_n \mathsf{U}_m \rangle$ influence the mean dynamics only relatively near the surface. This is so because, at heights such that $k_0 z$ is $O(1)$ or larger, the relative velocity $U + \omega_0/k_0$ is large compared with a typical turbulent velocity and the term $(\partial_t + U\partial_1)\langle \mathsf{U}_n \rangle$ and the pressure gradient dominate the mean flow equation (4).

(ii) In the turbulence-dominated region, the vertical scale of $\langle \mathbf{U} \rangle$ is small compared with the horizontal scale. It follows from this and the equation of continuity that $\langle \mathsf{U}_1 \rangle \gg \langle \mathsf{U}_3 \rangle$.

Certain assumptions are also made about the energetic components of the primary turbulent velocity $\mathbf{u}$.

(iii) The turbulent velocity components are of comparable magnitude in all three directions.

(iv) The x_1 scale of the turbulent components is long compared with the vertical scale, which is assumed to be somewhat less than $O(z)$. This latter assumption receives some support from the experimental results discussed in §4.

A straightforward application of restrictions (ii)–(iv) simplifies (6), the equation of motion for the turbulent perturbations, to

$$L_{nm} \hat{\mathsf{W}}_m = -\delta_{1n} u_3 \partial_3 \langle \mathsf{U}_1 \rangle. \tag{8a}$$

The interaction of the mean flow and the turbulence has been simplified to consideration of vertical advection of horizontal mean momentum by the turbulent vertical velocity. These equations are most useful when transformed to Fourier space, where they can be collapsed to a single differential equation for each frequency-wavenumber $\mathbf{k}$. This is done by introducing (7) (and an equivalent representation for the pressure) into (8a), multiplying the x_1 component by k_1/k (here $k^2 = k_1^2 + k_2^2$), then multiplying the x_2 equation by k_2/k and adding these two. Subtracting the x_3 equation multiplied by k from the x_3 derivative of the combined x_1, x_2 equation then yields

$$\left[i(\omega + Uk_1)\left(\frac{d^2}{dz^2} - k^2\right) - ik_1 U'' - \nu\left(\frac{d^2}{dz^2} - k^2\right)^2\right]\hat{\psi}(\mathbf{k}) = \hat{M}\hat{\psi}(\mathbf{k})$$

$$= -\frac{k_1}{k}\frac{d}{dz}\sum_{\mathbf{l}=\pm\mathbf{k}_0} v_3(\mathbf{k}-\mathbf{l})\frac{d}{dz}\langle \mathsf{V}_1(\mathbf{l}) \rangle, \tag{8b}$$

where ψ is a pseudo-stream function defined by

$$\frac{k_1}{k}\hat{\mathsf{V}}_1(\mathbf{k}) + \frac{k_2}{k}\hat{\mathsf{V}}_2(\mathbf{k}) = \frac{d\psi(\mathbf{k})}{dz}, \quad \hat{\mathsf{V}}_3(\mathbf{k}) = -ik\psi(\mathbf{k}).$$

It will be recognized that the operator $\hat{M}$ in (8*b*) is the Orr–Sommerfeld operator, which plays a central role in the theory of the stability of parallel laminar shear flows. As shown by Lin (1955), in the limit $\nu \to 0$ the solution of the Orr–Sommerfeld equation can be approximated by the solution of the inviscid equation obtained by setting $\nu = 0$ and taking the frequency ω, which is actually real, to be complex with a vanishingly small negative imaginary part. The approach employed here is to find ψ by setting $\nu = 0$ in (8*b*) and replacing ω in that equation by $\omega - i\mu$, where μ is an infinitesimal positive number which later in the analysis will be allowed to vanish. This procedure reduces (8*b*) to

$$\left[\Omega(\mathbf{k})\left(\frac{d^2}{dz^2}-k^2\right)-\Omega''(\mathbf{k})\right]\hat{\psi}(\mathbf{k}) = M\hat{\psi}(\mathbf{k}) = \frac{dR}{dz}(\mathbf{k}), \tag{9}$$

where $R(\mathbf{k}) = -\dfrac{k_1}{k}\sum_{\mathbf{l}=\pm\mathbf{k}_0} v_3(\mathbf{k}-\mathbf{l})\dfrac{d}{dz}\langle \mathsf{V}_1(\mathbf{l})\rangle$ and $\Omega(\mathbf{k}) = i\omega + iUk_1 + \mu$.

Turning now to the mean momentum equation (4), it follows from the basic long-wave assumption (i) that the right-hand side can be simplified to

$$D_{nm}\partial_3 \mathsf{R}_{m3} = -D_{nm}\partial_3(\langle u_3\hat{\mathsf{U}}_m\rangle + \langle u_m\hat{\mathsf{U}}_3\rangle).$$

Since by definition the mean flow is independent of x_2, the components $\langle u_2\hat{\mathsf{U}}_3\rangle$ and $\langle u_3\hat{\mathsf{U}}_2\rangle$ vanish and the remaining components can be derived from ψ. To make this specific, consider the computation of $\langle u_3\hat{\mathsf{U}}_1\rangle$, which, it will be seen, plays the dominant role in the mean flow dynamics. Clearly,

$$\langle -u_3\mathsf{U}_1\rangle = \sum_{\mathbf{l}=\pm\mathbf{k}_0}\tau(\mathbf{l})\exp\{i\mathbf{l}\,.\,\mathbf{s}\},$$

where
$$\tau(\mathbf{l}) = \sum_{\mathbf{k}}\langle -v_3(\mathbf{l}-\mathbf{k})\mathsf{V}_1(\mathbf{k})\rangle = \sum_{\mathbf{k}}\left\langle -v_3(\mathbf{l}-k)\frac{k}{k_1}\frac{d\psi(\mathbf{k})}{dz}\right\rangle. \tag{10a}$$

In general, the inverse of the operator M in (9) will be an integral operator, so that, letting z_1 denote the active or integration variable,

$$\frac{d\psi(\mathbf{k})}{dz} = \frac{d}{dz}M_{\mathbf{k}}^{-1}\frac{d}{dz_1}R(\mathbf{k},z_1) = N_{\mathbf{k}}\frac{d}{dz}R(\mathbf{k},z_1),$$

and, substituting the definition of R,

$$\tau(\mathbf{l},z) = \sum_{\mathbf{k}} N_{\mathbf{k}}\frac{d}{dz_1}\langle v_3(\mathbf{l}-\mathbf{k},z)\,v_3(\mathbf{k}-\mathbf{l},z_1)\rangle\frac{d\langle \mathsf{V}_1(\mathbf{l})\rangle}{dz_1}. \tag{10b}$$

This result is the Fourier analogue of (5) presented in the discussion of the general method. Certain features of the physics of Reynolds stress generation can be seen directly from this result. (i) As will be shown in §4 the primary turbulence modes v_n tend to be energetic only relatively near their critical heights, where $\omega + Uk_1$ is small. Consequently, the correlation $\langle v_n(-\mathbf{k},z)\,v_m(\mathbf{k},z_1)\rangle$ will be small unless $z-z_1$ and $\omega+Uk_1$ are both small. Thus, from (10*b*) it follows that the Reynolds stress at a height z is primarily determined by the mean flow near the same elevation. (ii) Since we are concerned only with levels at which $k_0 z$ is small, it is known that the energy of the primary turbulence is concentrated near

$k_1 z = O(1)$; it follows that the major contribution to (10*a*) comes from those $\psi(\mathbf{k})$ with values of k large compared with k_0. Hence the major contribution to $\tau(\mathbf{k}_0)$ comes from the interaction of $v_3(\mathbf{k}_0 - \mathbf{k})$ and $\psi(\mathbf{k})$ with $\mathbf{k} \simeq \mathbf{k} - \mathbf{k}_0$. Since $v(\mathbf{k})$ is concentrated near $\omega + Uk_1 = 0$, it follows that the important components of $\psi(\mathbf{k})$ are also associated with small values of $\omega + Uk_1$.

It is known from the theory of hydrodynamic stability that the operator M in (9) is nearly singular at the critical point. Consequently, the functions $\psi(\mathbf{k})$ vary rapidly near that point and, in fact, as $\mu \to 0$ the z derivative of ψ becomes infinite at the singular point and is large in that neighbourhood. As observed above, the nature of the primary turbulence correlation in (10*b*) dictates that it is the behaviour of $\psi(\mathbf{k})$ near its critical level that plays the main role in generating the turbulent stresses. Since in this region $\psi'(\mathbf{k}) \gg k\psi(\mathbf{k})$, and of the stresses relevant to the mean dynamics of a long-wave perturbation only $\langle u_3 \mathsf{U}_1 \rangle$ involves this quantity, we are led to simplify the mean momentum equation (4) to

$$L_{rm}\langle \mathsf{W}_m \rangle \simeq \delta_{1n}\partial_3 \mathsf{R}_{13} \simeq \delta_{1n}\partial_3\langle -u_3 \mathsf{U}_1 \rangle. \tag{11}$$

It can be seen by deriving the expressions analogous to (10*a*) for the other stresses that this involves neglecting $O(u\psi/z)$ and retaining $O(u\, d\psi/dz)$.

It remains now to develop a computationally useful inverse to M and use this inverse to compute $\langle u_3 \mathsf{U}_1 \rangle$ from (10*b*). It has already been mentioned that the important components of $\psi(\mathbf{k})$ will have their critical heights at values of z $O(1/k_1)$ or slightly less. In the critical region $\psi' \gg k\psi$; consequently (9) can be approximated by

$$\left[\Omega(\mathbf{k})\frac{d^2}{dz^2} - \Omega''(\mathbf{k})\right]\psi(\mathbf{k}) = \frac{dR(\mathbf{k})}{dz}.$$

It is reasonable that below the critical level, where $kz < 1$, this equation will apply. Because the viscous stresses have been neglected, the boundary condition on ψ' at $z = 0$ must be relaxed, but $\psi = 0$ at $z = 0$ still applies. There is presumably a region in the viscous sublayer where viscous effects rapidly bring ψ' to zero, but this region is outside our range of consideration. The general solution appropriate to the single boundary condition at $z = 0$ is

$$\psi(\mathbf{k}) = \Omega(\mathbf{k}, z)\int_0^z [\Omega^{-2}(\mathbf{k}, z_1) R(\mathbf{k}, z_1) + A_0]\, dz_1. \tag{12}$$

To establish the constant A_0 it is necessary to establish the behaviour of ψ as $z \to \infty$. This is most easily accomplished by noting that as z becomes large the appropriate scale of variation is $1/k$, and, in terms of the stretched co-ordinate $\eta = kz$, (9) becomes

$$\left[\ln\left(\frac{\eta}{kz_c}\right)\left(\frac{d^2}{d\eta^2} - 1\right) + \frac{1}{\eta^2}\right]\psi(k) = R'/iU_0 kk_1,$$

where μ has been neglected in Ω, and z_c is the critical height of the frequency wavenumber $\mathbf{k}$. As z increases above the value z_c, the ratio $\ln(\eta/kz_c)\,\eta^2$ becomes large, and an approximation to ψ can be obtained by neglecting the term $1/\eta^2$. The appropriate solution which vanishes as $\eta \to \infty$ is then

$$\psi = \frac{1}{iU_0 k k_1} \int_\infty^\eta \cosh(\eta - \eta_1) \frac{R(\eta_1)}{\ln(\eta_1/kz_c)} d\eta_1 + A_1 e^{-\eta}.$$

In the limit $\eta \to 0$, this approaches

$$\int_0^z \Omega^{-1}(z_1) R(z_1)\, dz_1 + A_2.$$

The inner solution matches with this outer solution if $A_0 = A_2 = 0$, as is easily seen by integrating the inner solution, with $A_0 = 0$, by parts to give

$$\psi = \int_0^z dz_1 \left[\Omega^{-1}(z_1) R(z_1) + \Omega'(z)\, \Omega^{-1}(z) \int_0^{z_1} dz_2\, \Omega^{-1}(z_2) R(z_2) \right] + O(\Omega^{-3}),$$

which clearly matches the outer solution as $\Omega \to \infty$.

Since in computing the turbulent stress it is the behaviour of ψ' in the inner region that is important, it is the integral operator in (12) that may be identified as M^{-1} in (10). Thus the perturbation shear stress is determined by

$$\tau(\mathbf{l}) = \sum_{\mathbf{k}} \left\langle v_3(\mathbf{l}-\mathbf{k}) \frac{d}{dz} \Omega(\mathbf{k}) \int_0^z \Omega^{-2}(\mathbf{k}, z_1)\, v_3(\mathbf{k}-\mathbf{l}, z_1) \frac{d\langle \mathsf{V}_1(\mathbf{l})\rangle}{dz_1} dz_1 \right\rangle.$$

A somewhat more compact representation can be given in terms of the wavenumber–frequency cross-spectrum of u_3. Thus the value of $\tau(\mathbf{l})$ at height z is

$$\tau(\mathbf{l}, z) = D_0(\mathbf{l}, z) \frac{d\langle \mathsf{V}_1(\mathbf{l})\rangle}{dz} + \int_0^z D_1(\mathbf{l}, z, z_2) \frac{d\langle \mathsf{V}_1(\mathbf{l})\rangle}{dz_2} dz_2, \qquad (13a)$$

where

$$D_0(\mathbf{l}, z) = \int d\mathbf{k}\, \Omega^{-1}(\mathbf{k}+\mathbf{l}, z)\, S_2(\mathbf{k}, z, z), \qquad (13b)$$

$$D_1(\mathbf{l}, z, z_2) = \int d\mathbf{k}\, iU'(z)\,(k_1 + l_1)\, \Omega^{-2}(\mathbf{k}+\mathbf{l}, z_2)\, S_2(\mathbf{k}, z, z_2), \qquad (13c)$$

$$S_2(\mathbf{k}, z, z_2) = \int \lim_{k_2\, \Delta k \to 0} \frac{\langle v_3(-\mathbf{k}, z)\, v_3(\mathbf{k}, z_2)\rangle}{\Delta\omega\, \Delta k_1\, \Delta k_2} dk_2. \qquad (13d)$$

S_2 is the cross-spectrum with respect to frequency and downstream wavenumber of the vertical velocity at two heights. The integrals over $d\mathbf{k}$ include only the k_1, ω plane.

4. The constitutive relation

Equation (13a) is a constitutive relation connecting $\tau(\mathbf{l})$, the Fourier amplitude of the turbulent shear stress R_{13}, to $\langle \mathsf{V}_1\rangle$, the Fourier amplitude of the mean velocity perturbation $\langle \mathsf{U}_1\rangle$. The D_0 term suggests the eddy viscosity law of Hussain & Reynolds (1972b); as it turns out, D_0 is generally a complex-valued function and is, therefore, more properly compared with an eddy visco-elasticity of the general type suggested by Davis (1972). The D_1 term is not so easily characterized, but, as is required by the long-wave approximation employed here, this term is insignificant for long perturbations.

The constitutive parameters D_0 and D_1 are weighted integrals of S_2, the spectrum of the undisturbed turbulence, and involve no adjustable parameters. But obtaining quantitative estimates of the constitutive parameters requires a rather extensive knowledge of the statistics of the undisturbed flow, and the required observational data are not presently available. What follows here is an attempt to estimate the structure of S_2 from available data and theoretical concepts. The aim is to predict D_0 and compare the prediction with experimental evidence on perturbation of turbulent shear flows, particularly the well-established law of the wall for constant stress boundary layers. Unfortunately, the uncertainties associated with estimating S_2 are too great to allow this to be a conclusive test but, in principle at least, this procedure would impose a stringent test of the theory. It is encouraging that the predicted D_0 is not inconsistent with observation.

On dimensional grounds, it can be argued that the energy containing portion of the spectrum S_2 is of the form

$$S_2(\mathbf{k}, z, z_2) = \frac{\langle u_3^2 \rangle z^2}{U} K(k_1 z)\, F\left(\frac{\omega z}{U} + k_1 z, \frac{U}{U_0}, k_1 z\right) R(k_1[z - z_2]), \tag{14}$$

where R also depends on all possible dimensionless groups but is defined such that $R(0) = 1$. The dimensional factor is dictated by the requirement that the integral of S_2 over ω and k_1 be equal to the variance $\langle u_3^2 \rangle$, which is independent of z.

In evaluating the integrals (13*b*, *c*), it is convenient to have K and F expressed in terms of functions that can be integrated analytically; for this purpose the simple forms

$$K = \frac{\alpha}{\pi}[\alpha^2 + (k_1 z)^2]^{-1}, \quad F = \frac{\beta}{\pi}[\beta^2 + (\omega z/U + k_1 z)^2]^{-1} \tag{15a,b}$$

have been chosen. These correspond to a downstream wavenumber spectrum which begins to fall off at $k_1 = \alpha/z$ and a frequency spectrum centred around the frequency $\omega = Uk_1$ and having a bandwidth $\Delta\omega = \beta U/z$. They are clearly inadequate approximations, but the major errors in the results appear to result from uncertainties in the data from which α and β are estimated rather than the inaccuracies inherent in the functional forms.

Most experimental observations of the turbulent fluctuations in constant stress boundary layers are reported in terms of pseudo-wavenumber spectra obtained by accepting Taylor's (1938) hypothesis that the turbulence is 'frozen' into the mean flow, and consequently obeys the dispersion relation $\omega = -Uk_1$. This is equivalent to assuming F to be a delta function centred at $\omega = -Uk_1$ and corresponds in (15*b*) to letting β approach zero. But of course, Taylor's hypothesis is not strictly correct, and, while the energy distribution is concentrated near $\omega = -Uk_1$, the distribution has a finite bandwidth $\Delta\omega$. This bandwidth plays a crucial role in determining the nature of the constitutive law (13).

As discussed by Lumley (1965), there are three reasons why Taylor's hypothesis should not be strictly correct. (i) The mean convection velocity U varies with z, and consequently, if a component contributes to the energy at more than one height, it follows that ω cannot equal $-Uk_1$ at all points. It seems likely that

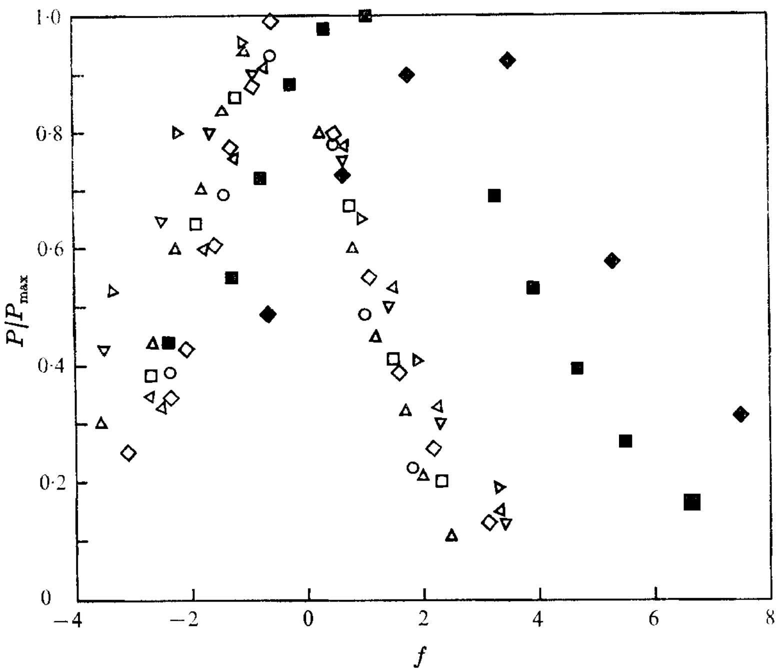

FIGURE 1. Morrison & Kronauer's spectrum $P(\omega, k)$ against the frequency parameter $f = (\omega + Uk_1)/(U_0 k_1)$. P normalized by an estimate of its maximum.

	$k_1 z$		$k_1 z$		$k_1 z$	y^+
◆	0·0044	■	0·015	◇	0·15	14
▷	0·022	▽	0·073	△	0·73	73
◁	0·059	□	0·2	○	210·0	198

this will lead to a $\Delta\omega$ of $O(U')$. (ii) It is likely that a high wavenumber mode will be convected by large turbulent components as well as the mean velocity. This would be expected to lead to a frequency broadening of $O(U_0 k_1)$. (iii) The turbulent modes cannot be expected to be perfectly frozen, and will have some time dependence when viewed in a frame moving with the local velocity. The magnitude of this time variation is not easily estimated, but it seems likely it will lead to frequency variations of approximately the same order as the two other effects.

Morrison & Kronauer (1969) measured the spectrum $P(\omega, k_1)$ of the turbulent component u_1 in a fully developed pipe flow. While this is not precisely the quantity required here, it seems likely that their spectra should be similar in general structure to $S_2(\omega, k_1, z, z)$. Their data are unique, in that it is possible to estimate from them the general form of F that determines the frequency dependence. From their published curves, I have constructed several cross-sections of $P(\omega, k_1)$ at fixed k_1. This was accomplished from a manuscript which contained full page versions of their figures 7 (p)–(r). The results, normalized to the maximum value of P at the selected value of k_1 and z, are presented in figure 1. The data cover values of $k_1 z$ between 0·0044 and 2·0. From these curves it is evident that, for $k_1 z > 0{\cdot}1$, the energy distribution is concentrated near $\omega = -Uk_1$, and that the

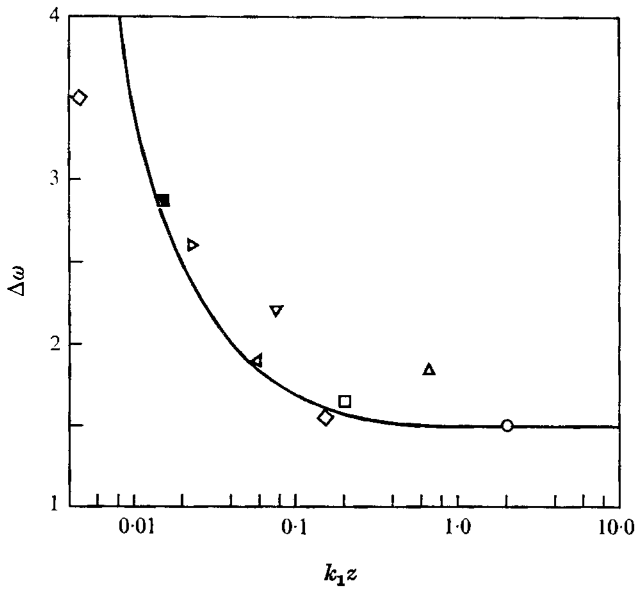

FIGURE 2. Bandwidth $\Delta\omega$ from Morrison & Kronauer against k_1z. Symbols as in figure 1.

bandwidth is proportional to U_0k_1, as Lumley suggests. It is also evident that, as k_1z becomes small, the bandwidth decreases less slowly than U_0k_1, which is consistent with the suggestion that, for small k_1, the bandwidth $\Delta\omega$ is proportional to U'. It appears plausible, then, to choose $\beta = (U_0/U)(\beta_0+\beta_1|k_1z|)$, which corresponds to a spectral density decrease of one half at the frequencies

$$\omega = -Uk_1 \pm \Delta\omega, \quad \text{where} \quad \Delta\omega = \beta_0 U' + \beta_1|U_0k_1|.$$

From the Morrison & Kronauer data shown in figure 1, the bandwidth $\Delta\omega$ was estimated and plotted in figure 2. The plotted curve corresponds to $\beta_0 = 0{\cdot}02$ and $\beta_1 = 1{\cdot}5$; clearly these estimates are subject to considerable uncertainty.

The frequency spectrum of u_3 has been measured in the atmospheric boundary layer, where the Reynolds number is large and the constant stress layer is thick, by Volkov (1969) and Miyake, Stewart & Burling (1970). These measurements were taken at elevations where $U_0/U \ll 1$, so that Taylor's hypothesis is valid and (14) corresponds to

$$S(\omega) = \int dk_1 S_2(\mathbf{k},z,z) = \langle u_3^2\rangle \frac{z}{U}\frac{\alpha}{\pi}[\alpha^2+(\omega z/U)^2]^{-1}.$$

The mean frequency spectra reported by these workers are presented in figure 3, along with the corresponding curve computed using $\alpha = 1{\cdot}2$. While the chosen functional form is clearly inadequate, the uncertainty introduced in the results is small compared with that arising from estimating the frequency function F.

The function $R(k_1[z-z_2])$ appearing in (14) is impossible to estimate from any data. However, R enters only into the computation of D_1, and it can be shown that this coefficient is negligible so long as R is nowhere large compared with $R(0)$, a supposition which appears reasonable.

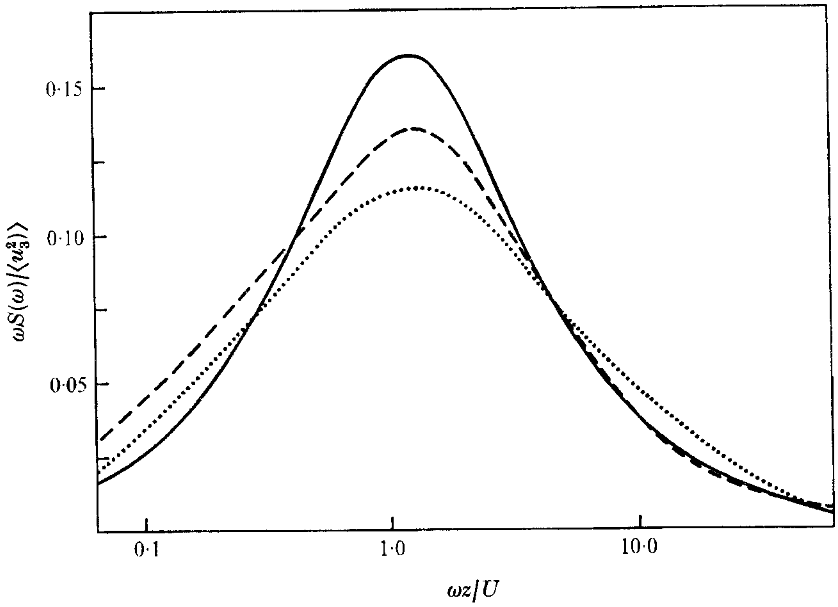

FIGURE 3. Frequency spectrum $S(\omega)$ normalized by $\langle u_3^2\rangle/\omega$. ——, equation (14); ······, Miyake *et al.*; – –, Volkov.

Having specified the cross-spectrum S_2, it is now possible to evaluate the constitutive parameters D_0 and D_1 directly from (13). The details of this computation are given in the appendix, where it is shown that

$$D_0(\mathbf{k}_0, z) = \frac{\langle u_3^2\rangle}{U^1}[\gamma^2 + (\alpha\beta_1)^2]^{-1}\left[\gamma + \frac{2\alpha\beta_1}{\pi}\ln\frac{\alpha\beta_1}{\gamma}\right], \tag{16}$$

where $\gamma = \beta_0 + (\omega_0 + k_0)/U'$ and the imaginary part of the logarithm is taken between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. It is also shown in the appendix that, for the long-wave perturbations considered here, the stress associated with D_1 is negligible (as it must be if the long-wave approximation is to remain valid), and therefore the constitutive equation may be approximated by

$$\tau(\mathbf{k}_0, z) = D_0(\mathbf{k}_0, z)\frac{d}{dz}\langle \mathbf{V}_1(\mathbf{k}_0, z)\rangle. \tag{17}$$

5. Conclusions

The analysis has led to a quantitative prediction of the turbulent shear stress generated by long-wave perturbations of a constant stress boundary layer. The approach is based on specific statistical hypotheses and mathematical approximations rather than phenomenology. Consequently, certain features of the analysis are testable by more exact solution of the dynamical equations, and, since the model involves no adjustable parameters, comparison with measurements could, in principle at least, provide a stringent test of validity. For example, the approximate solution of the Orr–Sommerfeld equation (8) can be tested through extensive, but straightforward, numerical solution. The long-wave approximation is similarly testable by evaluation of the terms neglected on the right-hand side of (8) and numerical solution of that equation. Some verification of this approximation is provided by the demonstration in the appendix that the stresses associated with D_1 are negligible.

The 'statistically small' assumptions are the only ones essential to the method and are apparently the most dubious. They are not easily verified except by comparison of predicted and measured turbulent stresses. The linearization assumptions (2*a*) do receive some *a posteriori* support from the fact that the analysis based on these assumptions results in perturbations that are regular functions of the boundary perturbation amplitude, and consequently obey the linearization assumptions when the perturbation amplitude is small. The quasi-Gaussian approximation (2*b*) is not so easily justified, except in the physically uninteresting case of weak turbulence. Hence the present analysis must be considered only the first term in an expansion about the state of no turbulence.

In view of the limitations of the analysis it is encouraging that it produces a predicted stress against mean velocity relation which is not inconsistent with the one well-established experimental observation, namely the 'law of the wall'. According to this law, a small mean flow perturbation $\langle \mathsf{U}_1\rangle = \epsilon \ln (z/z_0)$ of a boundary layer with the undisturbed shear stress $\langle -u_1 u_3\rangle$ results in a perturbation shear stress τ given by

$$\tau = 2\kappa\langle -u_1 u_3\rangle^{\frac{1}{2}} z \frac{d}{dz}\langle \mathsf{U}_1\rangle, \tag{18}$$

where κ is von Kármán's constant. This particular perturbation corresponds in the theory to $k_0 = \omega_0 = 0$, and the perturbation stress should then be given by (17) with D_0 computed from (16) when $\gamma = \beta_0$. This results in a predicted relationship between τ and $\langle \mathsf{U}_1\rangle$ which is identical with (18), except that the constant κ is replaced by

$$\kappa = [2U_0]^{-1}\frac{\langle u_3^2\rangle}{\langle -u_1u_3\rangle^{\frac{1}{2}}}[\beta_0^2 + (\alpha\beta_1)^2]^{-1}\left[\beta_0 + \frac{2\alpha\beta_1}{\pi}\ln\frac{\alpha\beta_1}{\beta_0}\right].$$

Volkov (1969) reported a vertical normal stress to shear stress ratio of 1·4, while Miyake, Stewart & Burling (1970) reported ratios ranging from 1·5 to 3·5. It is commonly accepted that $U_0 \simeq 2{\cdot}5\langle -u_1u_3\rangle^{\frac{1}{2}}$. Using these values and the estimates $\alpha\beta_1 = 1{\cdot}8$ and $\beta_0 = 0{\cdot}02$ from §4 (which were arrived at before attempting this comparison) leads to a predicted value of κ between 0·45 and 1·1, which is to be contrasted with the correct value of 0·4.

In view of the large uncertainties associated with the estimated form for S_2, the fact that the theory is not inconsistent with observation must, to a large extent, be considered fortuitous. The model functions for F and K used in (14) were chosen primarily for convenience, and no estimate has been made of the sensitivity of D_0 to the particular function chosen. The available data were stressed very heavily in determining the parameters α, β_0 and β_1, and the ratio $\langle u_3^2\rangle/\langle -u_1u_3\rangle$ is not well established experimentally. For example, the adjusted values $\alpha\beta_1 = 2{\cdot}1$ and $\beta_0 = 0{\cdot}02$ coupled with a stress ratio of 1·5, all of which are consistent with the data, lead to exact agreement with the law of the wall; similarly, small adjustments lead to predictions that are unacceptably different

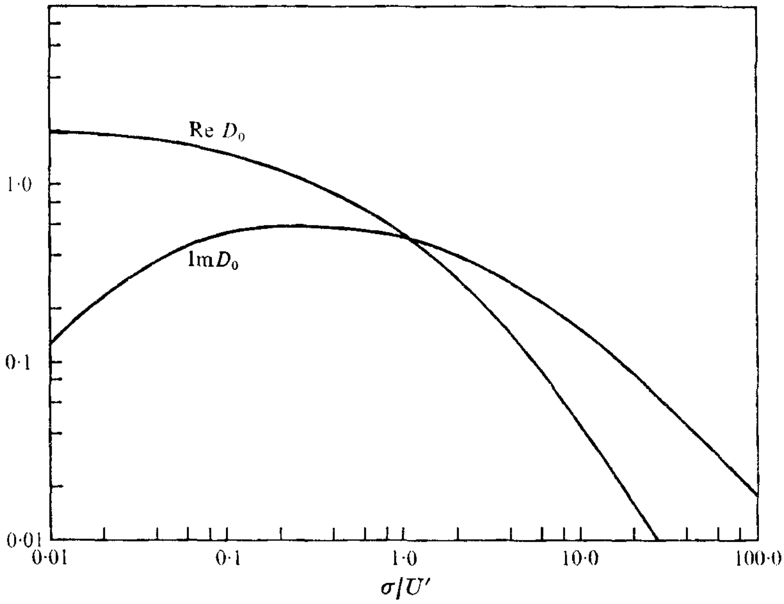

FIGURE 4. Eddy visco-elasticity D_0 against $\sigma = \omega + Uk_1$. The ordinate is $D_0 U'/\langle -u_1 u_3 \rangle$.

from observation. The particular form of S_2 chosen is not essential to arriving at an estimate of D_0, and, when more extensive data are available, it will be worth while to recompute this parameter directly from (13) using a more accurate model. But, for the present, when the available data are so minimal this would be inappropriate.

But to a large extent the question of precise numerical agreement of theory and experiment misses the mark. It seems to me that the most important feature of this analysis is that it provides a dynamical explanation for the success of the phenomenologically motivated eddy viscosity models of turbulence and may provide a rudimentary framework for developing useful empirical constitutive laws for perturbed turbulent flows. In this context it is instructive to examine the general behaviour of the eddy visco-elasticity D_0 as a function of the intrinsic frequency $\sigma = \omega + Uk_1$. The real and imaginary parts of D_0 are plotted in figure 4 using the adjusted values of $\alpha\beta_1$, β_0 and the stress ratio. The figure shows only positive intrinsic frequencies, since $D_0(\sigma)$ is the complex conjugate of $D_0(-\sigma)$.

From the general behaviour of D_0 one can see a possible explanation of the success of the eddy viscosity models of Hussain & Reynolds (1972*b*) and Davis (1972). Both of these studies employed a constitutive relation which for long waves is approximated by (17) with $D_0 = 2\kappa\langle -u_1 u_3\rangle^{\frac{1}{2}} z$ in the constant stress portion of the flow. From figure 4 it is evident that, for $|\omega + Uk_1| < 0{\cdot}1U'$, this is a good approximation to the theoretical value. One expects that the turbulent stresses are of secondary importance, except near the critical layer where $\omega + Uk_1$ vanishes and near boundaries where U' becomes large. Consequently one might expect that, even though the constitutive relation is not strictly viscous, neglect of the elasticity associated with the imaginary part of D_0 would not greatly influence the dynamics of the mean flow perturbations. The fact that Davis (1972) obtained similar predictions of surface pressure for flow over a wavy boundary using this eddy viscosity model C and eddy visco-elasticity model D tends to corroborate this conjecture.

It is also of some interest to compare the present dynamical constitutive relation with the phenomenological eddy visco-elasticity model advanced by Davis (1972). That model was based on the conjecture that the perturbation turbulent stress is determined by the recent history of rate of strain experienced by a fluid parcel moving at the mean flow velocity. In the case of long-wave perturbations, this leads to a relation for the perturbation turbulent shear stress of the form

$$\langle u_1 \mathsf{U}_3 + u_3 \mathsf{U}_1 \rangle (\mathbf{x}, t) = \int_{-\infty}^{0} -H(\tau)\, \partial_z \langle \mathsf{U}_1(\mathbf{x} + \hat{\mathbf{x}}_1 U\tau, t+\tau) \rangle \, d\tau,$$

where $\hat{\mathbf{x}}_1$ is a unit vector and the memory function H must vanish as $\tau \to -\infty$ if the constitutive relation is not purely elastic. This form of phenomenological model leads to the constitutive law (17) with D_0 given by

$$D_0(\sigma) = \int_{-\infty}^{0} H(\tau) \exp\{i\sigma\tau\}\, d\tau,$$

where σ is the intrinsic frequency $\omega + Uk$. Some manipulation shows that the form of D_0 given in (16) leads to a memory function

$$H(t) = \frac{1}{\pi} \langle u_3^2 \rangle \{\exp U'\beta_0 t\} [\mathrm{Ci}(\lambda) \sin\lambda + \{\tfrac{1}{2}\pi - \mathrm{Si}(\lambda)\} \cos\lambda],$$

where Ci and Si are the cosine and sine integral functions discussed by Gautschi & Cahill (1964), and $\lambda = -U'\alpha\beta_1 t$.

While the exact form of the memory function associated with the dynamical theory is not of significance, it may be of interest in the development of phenomenological models to isolate the characteristic time scales of this function. Formally, there are two time scales, namely

$$\tau_1 = [U'\beta_0]^{-1} \quad \text{and} \quad \tau_2 = [U'\beta_1 \alpha]^{-1}.$$

The frequency $1/\tau_1$ will be recognized immediately as the limit as $k_1 \to 0$ of the frequency bandwidth $\Delta\omega$ of the spectrum S_2 discussed in §4. Noting that the parameter α is associated with the wavenumber k_c at which the wavenumber spectrum begins to fall off by $k_c = \alpha/z$, it follows that the frequency $1/\tau_2 = U_0 k_c \beta_1$ is the bandwidth $\Delta\omega$ at the wavenumber k_c. These observations suggest a sensitivity of the dynamics of turbulent stress generation to the details of the turbulent energy distribution near the dispersion relation $\omega = -Uk_1$, which has not previously been noted.

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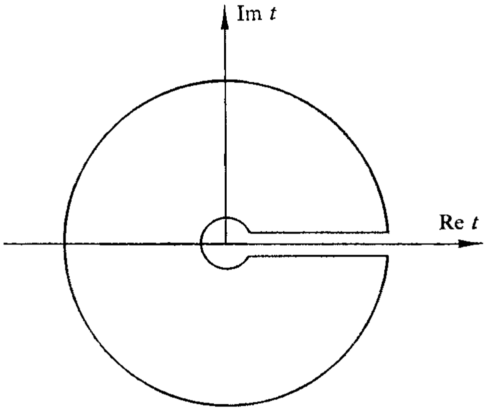

FIGURE 5. Contour used to evaluate integrals (A 1) and (A 2).

Appendix

With the cross-spectrum S_2 modelled by (14) and (15), it is possible to evaluate D_0 and D_1 directly from their definitions in (13). It is convenient first to evaluate the integral over ω using integration in the complex plane and the method of residues. The frequency function F contributes simple poles at

$$\omega + Uk_1 = \pm i\beta U/z;$$

the function Ω^{-1} contributes a pole in the upper half-plane (since μ is positive), which may be avoided by closing the contour from below. Carrying out the integrations, then letting μ go to zero, yields

$$D_0(\mathbf{1}, z) = \frac{\langle u_3^2\rangle z}{U(z)}\frac{\alpha}{\pi}\int\frac{dx}{\alpha^2+x^2}\frac{1}{\beta+i(x_0+y_0)},$$

$$D_1(\mathbf{1}, z, z_2) = \frac{\langle u_3^2\rangle z_2}{U(z_2)}\frac{\alpha}{\pi}\int\frac{d\hat{x}}{\alpha^2+\hat{x}^2}R\left(\hat{x}\left[1-\frac{z}{z_2}\right]\right)\frac{iU'(z)(\hat{x}+\hat{x}_0)/U(z_2)}{[\hat{\beta}+i(\hat{x}_0+\hat{y}_0)]^2},$$

where $x = k_1 z$, $y = \omega z/U(z)$, a caret denotes evaluation at z_2, and the subscript 0 denotes quantities involving ω_0, k_0 in place of ω, k_1. Because of the expected decay of R away from the point $z = z_2$, $D_1(\mathbf{1}, z, z_2) < D_1(\mathbf{1}, z, z)$ and this latter quantity will soon be found negligible for long waves. Substituting the form for β and noting the symmetry of the integrands about $x = 0$ converts these expressions to

$$D_0(\mathbf{k}_0, z) = \frac{\langle u_3^2\rangle}{U'}\int_0^\infty\frac{2\alpha}{\pi}\frac{dx}{\alpha^2+x^2}\frac{1}{\beta_1 x+\gamma}, \tag{A 1}$$

$$D_1(\mathbf{k}_0, z, z) = \frac{\langle u_3^2\rangle}{zU'}x_0\int_0^\infty\frac{2\alpha}{\pi}\frac{dx}{\alpha^2+x^2}\frac{i}{[\beta_1 x+\gamma]^2}, \tag{A 2}$$

where $\gamma = \beta_0 + i(\omega_0 + Uk_0)/U'$. In the context of the long-wave approximation, $x_0 = k_0 z$ is a small number; consequently the term in (13a) associated with D_1 should be negligible. From (A 1) and (A 2) it follows that D_1/D_0 is

$O(U'kz/U_0) = O(k)$; consequently the D_1 term in (13*a*) is negligible so long as $kz \ll 1$ and the vertical scale of $\langle \mathsf{U}_1 \rangle$ is not large compared with z.

The above integral determining D_0 is easily evaluated when it is noted that

$$\int_0^\infty Q(x)\,dx = \frac{-1}{2i}\int_c \ln(t)\,Q(t)dt,$$

where c is the contour depicted in figure 5. This results in the relatively simple expression for D_0 given in (16).

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